A haptic rendering robot (Haptic Master, Moog, Inc.) is used in experiments to understand how humans control interaction with objects. The robot generates a relation between subjects’ applied forces and the robot’s resulting motion. That relation may be programmed to render virtual objects. Under certain conditions the system may exhibit coupled instability: when its handle is grasped by a human, the robot becomes unstable, breaking into undesirable oscillations. You are to analyze that phenomenon.

The robot control system is depicted in Figure 1. It consists of an “inner” motion controller that accurately reproduces motion commands. The motion commands are derived from a virtual model of the object to be rendered, in response to forces applied by the human subject. Applied forces are measured by a transducer mounted in the handle on the end of the robot.

1. Coupled instability may occur when the robot renders a freely-moving rigid body of mass $M$. Develop a mathematical model and analysis to explain the coupled instability.

For simplicity:

- Consider motion in a single degree of freedom.
- The motion controller may be approximated as having a flat response to 20 Hz with a first-order roll-off at higher frequencies, i.e. $x(s) \approx \frac{\lambda}{s^2 + \lambda} x_{command}(s)$

---

1 As implemented, the motion commands (and feedback sensors) include position, velocity and acceleration (PVA in the figure) but that is a detail.
Fig. 2: Ball-and-cup task. Subjects interact with a HapticMaster (bottom right) simulating the dynamics of the ball and cup (bottom left). Targets, timing and ball and cup motions are displayed visually (top).

where $x$ is position of the handle, $s$ is the Laplace variable, $\lambda$ is a constant and $x_{\text{command}}$ is the commanded position of the handle.

- In implementation, rigid body motion is simulated to include a small constant amount of viscous damping, $B$.

2. Does your analysis predict instability for smaller or larger $M$?

For some values of $M$ the system exhibits unstable low-frequency oscillations even without contact with a subject, despite the fact that the “inner” loop motion controller is robustly stable.

3. What might account for that behavior?

In some experiments, subjects move a simulated cup with a ball rolling inside it (Figure 2.) They see the cup as a circular arc with a ball inside. The circular cup shape gives the ball the dynamics of a simple pendulum. The subject can only control the ball indirectly by applying forces to the cup. Object dynamics are simulated as a mass $M$ constrained to move horizontally in one dimension with a simple pendulum of mass $m$ and length $l$ suspended from it. The virtual model equations of motion are:

$$(M + m)\ddot{x} = ml(\dot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) + f$$

$$l\ddot{\theta} = \ddot{x} \cos \theta - g \sin \theta$$

where the overdot denotes time-differentiation, $\dddot{x}$ is the cup’s horizontal acceleration, $\theta$ is the ball’s angular displacement from the cup bottom, $f$ is the force exerted by the subject and $g$ is gravitational acceleration. A linearized model resembles two masses ($M$ and $m$) connected by a linear spring.

4. Is this simulation more or less prone to instability?