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Department of Mechanical Engineering
Massachusetts Institute of Technology
Doctoral Qualifying Examination
Systems and Control Written Exam Problem
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Show how you arrived at your answer.
Relax!

Consider a one-axis positioning system carrying a vertical arm sliding on a horizontal table, as shown in Figure 1. A DC motor fixed to the base frame generates driving force $F$ on the shoulder part of the arm link through a high-stiffness steel belt. The shoulder part sliding on the table is supported by a set of air bearings to eliminate friction. The air bearings, however, are limited in stiffness; the arm swings within the vertical plane when a horizontal force is applied to the tip of the arm. The rotational stiffness of the shoulder has been identified to be $Ka$, as shown in Figure 1.

For position control, an optical encoder was attached to the shoulder part in order to measure the position of the moving part at point A, i.e. coordinate $x$ in the figure. PD control using this encoder signal showed stable responses for a wide range of the control gains. For this control system a step response was measured at two different points. Figure 2 shows the time profile of position $x$ measured by the encoder at point A, while Figure 3 shows the one observed at point B, i.e. the tip position $y$. Note that a significant undershoot was observed at the arm tip.

Since the fundamental goal of this positioning system is to locate the end-effector at the tip of the arm, a new sensor was attached to the arm tip in order to close the loop based on the measured tip position. It turned out that PD control around the tip sensor was totally unstable; the system exhibited instability even for a small proportional gain. You are requested to build a simple mathematical model that elucidates the dynamic behavior described above and discuss stability.

The parameters provided in Figure 1 are:

$$
\begin{align*}
  m & \quad \text{total mass of the arm and shoulder} \\
  I & \quad \text{centroïdal moment of inertia} \\
  L_1 & \quad \text{distance between point A and the centroid} \\
  L_2 & \quad \text{distance between point B and the centroid} \\
  Ka & \quad \text{rotational stiffness of the shoulder supported by the air bearings}
\end{align*}
$$

If needed, you can use other parameters and variables, but make sure that the notation you use must be clearly defined.

1). Obtain the transfer function from driving force $F$ to position $x$ and the one from $F$ to $y$ that are of the lowest order and that are competent to portray the dynamic behavior described above.
2). Plot the root locus of the PD control system using the encoder at point A, and discuss stability. In plotting the root locus, choose appropriate parameter values that physically make sense.

3). Obtain conditions for the parameter values to exhibit the undershoot behavior, and discuss the physical sense of the conditions.

4). Plot a root locus of the PD control system using the tip sensor, and discuss stability.

Figure 1  Schematic of horizontal positioning system with air bearings

Figure 2  Step response of $x$

Figure 3  Step response of $y$