Qualifying Examination (Written)
Systems, Dynamics, and Control
Department of Mechanical Engineering
Massachusetts Institute of Technology
May 26, 2009

• This is a closed-book exam.
• There is one question with 6 parts in this exam. Each part carries equal weight.
• Please make sure that your exam has a total of 3 pages, including this cover page.
• All sections of this problem assume linearity and time-invariance.
• In your solutions, clearly show your approach and results.
It can be shown that a one degree of freedom conductive heat flow problem yields a transfer function of the form

\[ G(s) \equiv \frac{T(s)}{Q(s)} = \frac{1}{\sqrt{s}}. \]  \hspace{1cm} (1)

Be sure to note the square root dependence on \( s \), which is characteristic of diffusion processes.

Here, \( Q(t) \) is heat flow input to the system in units of Watts, and \( T(t) \) is the resulting output temperature in degrees Kelvin at the input point, and thus the transfer function has units of \( ^\circ K/W \). We assume that both variables can take positive and negative values. (In physical terms, this means that the analysis is conducted about some operating point, which can be ignored in this problem.) Further, \( Q(s) = \mathcal{L}\{Q(t)\} \) and \( T(s) = \mathcal{L}\{T(t)\} \) are the Laplace transforms of the corresponding time functions, and \( s \) is the Laplace complex frequency, as usual.

a) (10 points) Make a carefully drawn and dimensioned Bode plot for this transfer function, showing the magnitude and phase as a function of frequency \( \omega \) from 1 to 1000 rad/sec.

b) (10 points) This thermal transfer function is now embedded as the plant in a proportional control loop with controller gain \( K_p > 0 \) as shown below:

\[ \begin{array}{cc}
T_r & \\
\downarrow & \\
\mathbb{K}_p & \\
\downarrow & \\
Q_d & \\
\downarrow & \\
T & \\
\end{array} \]

Here, \( T_r(t) \) is the reference temperature input to the control loop, and \( Q_d(t) \) is an additive disturbance heat flow acting at the plant input. The temperature error at the output of the summing junction is defined as \( e(t) \). All these variables have corresponding Laplace transforms.

The answers to the following questions should be given in terms of the system parameters. You may assume that the loop is stable. Be sure to show your reasoning.

b1) Assume \( Q_d = 0 \) and that \( T_r = 1 \) for all time. What is the steady-state value of \( e \)?

b2) Assume \( T_r = 0 \) and that \( Q_d = 1 \) for all time. What is the steady-state value of \( e \)?

b3) Now assume that \( Q_d = 0 \) and that \( T_r(t) = \sin \omega t \) for all time. In this case, \( e(t) \) will take the form \( e(t) = M \sin(\omega t + \phi) \). What are the values of \( M \) and \( \phi \) in terms of \( K_p \) and \( \omega \)?

c) (10 points) Now, we set \( K_p = 10 \). What are the loop unity-gain crossover frequency \( \omega_c \) and corresponding phase margin \( \phi_m \)? Show your reasoning.
d) (10 points) We now add a time delay of $T_0$ seconds to the temperature measurement as shown below. For $K_p = 10$, what is the largest value of $T_0$ for which the loop is stable? Show your reasoning.

\[
\begin{align*}
\text{Tr} & \quad + \quad \circ \quad K_p \quad + \quad \times \quad G(s) \quad T \quad +
\end{align*}
\]

\[
\text{Time Delay of } \frac{T_0}{\text{seconds}}
\]

\[
\text{Qd}
\]

\[
\frac{Q}{Q}
\]

\[
\frac{T(s)}{Q(s)} = \frac{A}{s} \left( \frac{\alpha \tau_1 s + 1}{\tau_1 s + 1} \right) \left( \frac{\alpha \tau_2 s + 1}{\tau_2 s + 1} \right) \left( \frac{\alpha \tau_3 s + 1}{\tau_3 s + 1} \right)
\]

(2)

f) (10 points) Conduct a formal Nyquist test analysis for the system with the time delay value of $T_0$ above which puts the loop on the stability margin. That is, show the Nyquist D-contour and image contour in the case where $T_0$ is equal to the largest value for which the loop is stable. Also show us what the Nyquist plot looks like, and why the system will be stable for smaller values of $T_0$, and unstable for larger values of $T_0$. Be sure to indicate how you have handled the infinite gain of the plant at $s = 0$.

\[
G_{\text{app}}(s) = \frac{T(s)}{Q(s)} = \frac{A}{s} \left( \frac{\alpha \tau_1 s + 1}{\tau_1 s + 1} \right) \left( \frac{\alpha \tau_2 s + 1}{\tau_2 s + 1} \right) \left( \frac{\alpha \tau_3 s + 1}{\tau_3 s + 1} \right)
\]

Suppose that $\tau_1 = 1$ sec, $\tau_2 = 0.1$ sec, and $\tau_3 = 0.01$ sec. The unitless parameter $\alpha$, where $1 < \alpha < 10$, has the same value in all three terms, and $A$ is a gain parameter with units of degrees Kelvin per Watt. For the chosen time constants, plot the transfer function poles and zeros on the s-plane, with $\alpha$ as a parameter. For these time constants, what value of $\alpha$ will give the best approximation to the original transfer function $G(s)$? Why? For this optimum $\alpha$, what is the best value of the gain $A$? Show your reasoning.