A slender vertical beam, with length $L$ and constant bending stiffness $EI$, is subjected to a linearly varying tension $T(x) = T_0 + Wx$ (where $T_0$ is the tension at the bottom of the beam $x = 0$, and $W$ a constant weight per unit length). The mass of the beam per unit length is $m$ and the acceleration of gravity is $g$, so that $W = mg$. The beam is supported by pin joints at both ends and is undergoing small free transversal (bending) oscillations $u(x, t)$ in the horizontal direction.

a) Derive the general linear partial differential equation of motion and the associated boundary conditions, indicating your assumptions.

b) If we assume that $T_0 \gg WL > 0$ so that the tension $T(x)$ can be considered effectively constant along the length of the beam and equal to $T_*$, derive analytical expressions for the natural frequencies and mode shapes of the beam.