Adaptive Control of a Generic Hypersonic Vehicle

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Air-breathing Hypersonic Vehicles

- Fly at speeds exceeding Mach 5
- Do not require onboard oxidizer for combustion
- Show significant potential for
  - Transportation
  - Military
  - Space access

- Achieved nearly Mach 10 using SCRAMjet propulsion
- Demonstrator vehicle for high performance maneuvers
- Made nearly 200 flights achieving almost Mach 7
- Record breaking 4 minute SCRAMjet powered flight

North American X-15
Rocket powered

1959

NASA X-43
2001

AFRL Roadrunner GHV
2010

Boeing X-51
2014
How an Air-breathing Hypersonic Vehicle Works

- Supersonic airflow required for combustion
  - Rockets are used to attain Mach 3
  - SCRAMjet engine used to accelerate to Mach 5-10

Goal: Design controller for aggressive maneuvering
Outline

• Challenges
• Unstart modeling
• Problem Statement
• Adaptive Control Design
• Results
• Aircraft Model
  – Uncertainties: CG Shift, Control Effectiveness, Stability Derivatives
• Adaptive Control Design
• Results
Challenges
Overview

- Are highly open-loop **unstable**
- Contain significant **bias** in incidence angle measurements
- Are **difficult to model**
  - Poor CFD models
  - Limited/no **wind tunnel data**
  - Exhibit significant **engine-airframe coupling**
  - Complex **shock interactions**

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Controlling hypersonic vehicles is very challenging

- Must be able to operate over a **large flight envelope**
  - Largely varying **dynamic pressure**
  - Harsh and **uncertain environments**
Challenges
Model Uncertainties

Actuators have rate and magnitude saturation limits

CG location can change during flight

Control effectiveness can diminish during flight

Cannot obtain accurate stability derivatives from CFD or wind tunnel data

Challenge: models are uncertain
Introduction of Unstart
Exacerbates the Problem Further

What is \textbf{unstart}? 

- Complex shock structure exists around a hypersonic vehicle
- Shock interactions greatly impact the \textit{aerodynamic and propulsive characteristics} of vehicle

![Diagram of unstart phenomenon](image)

- Shock train propagates upstream
- \textbf{Normal shock in front of inlet}
- \textbf{Subsonic} flow through engine
- Engine experiences \textbf{unstart}
Unstart Modeling
The Effect of Unstart

When unstart occurs:

• Abrupt change in pressure distribution around vehicle
• Vehicle **forces** and **moments** change

- Significant additional pitch, roll, and yaw **moments**
- Loss of **lift and thrust**
- Increase in **drag**

**Challenge:** unstart destabilizes vehicle

Unstart Modeling

Causes of Unstart

- Causes of unstart:
  - Excessive combustion pressure
  - Shock-boundary layer interactions
  - Change in air capture ratio

\[
\frac{A_0}{A_1} \quad \text{Air capture ratio}
\]

Air capture characteristics change with **incidence angles** and **Mach number**

One mechanism of unstart:

\[ (\alpha, \beta, M) \in D \]

\[ \alpha, \beta: \text{incidence angles} \]
Problem Statement

Design a controller for the hypersonic vehicle that can:

• Accommodate high levels of **model uncertainty**
• Ensure satisfactory **tracking**
• Be robust to **delays**
• Achieve aggressive maneuvers
  • High **bank angle** turns
  • Large **angle of attack** climbs
• Avoid **unstart**

**My approach:** **adaptive control**
Consider the linear **uncertain** plant

\[
\dot{x} = Ax + B\Lambda u + B_{\text{ref}}z_{\text{cmd}}
\]

### Classical Baseline + Model Reference Adaptive Approach

- Design a robust linear **baseline controller**
- Include an **adaptive control component** which adjusts gains online in response to errors
- Error is generated by comparing the plant response to an ideal, or **target response**
- Uses **state feedback**

\[
u = (K_{\text{lqr}} + \theta(t))^\top x
\]

**Adaptive Design**

**Control Overview**
My Adaptive Controller
(To Avoid Unstart)

• Problem: **Corrupted sensors** make **state feedback** highly vulnerable
  - Desired sideslip angle is **zero**
  - Measured sideslip angle contains **bias**
  - Regulating measured sideslip to zero → sideslip angle nonzero

Consider lateral-directional dynamics

- **State:**
  - $\beta$ - Sideslip angle
  - $p$ - Roll rate
  - $q$ - Pitch rate
  - $\phi$ - Roll angle

My solution: **adaptive output feedback**
Avoiding Unstart
An Adaptive Approach

Design adaptive controller based on $y$

Plant:
$$\dot{x} = Ax + B\Lambda u + B_{\text{ref}}z_{\text{cmd}} \quad y = Cx$$

CRM:
$$\dot{x}_m = A_m x_m + B_{\text{ref}}z_{\text{cmd}} + L(y_m - y)$$

Adaptive output feedback:
$$u = (K_{1qr} + \theta(t))^\top x_m$$
$$\dot{\theta} = -\Gamma x_m (S_1 e_y)^\top \text{sign}(\Lambda)$$

Reference model provides target response
Adaptive Control
Closed-Loop Reference Model Overview

- **Open-loop** reference model
- **Large transients** with fixed target

**Closed-loop Reference Model (CRM)**
- Shapes the target
- **Leads to smoother transients, better robustness**

- **Closed-loop** reference model*
- **Smooth transients** with flexible target

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Avoiding Unstart
An Adaptive Approach

Design adaptive controller based on $y$

Plant:
$$\dot{x} = Ax + B\Lambda u + B_{\text{ref}}z_{\text{cmd}} \quad y = Cx$$

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Adaptive output feedback:
$$u = (K_{\text{lqr}} + \theta(t))^\top x_m$$
$$\dot{\theta} = -\Gamma x_m (S_1 e_y)^\top \text{sign}(\Lambda)$$

$S_1$ and $L$ need to be chosen to guarantee stability
Avoiding Unstart
An Adaptive Approach

Tracking errors: 
\[ e_x = x - x_m \quad e_y = Ce_x \quad e_s = S_1Ce_x \]

Error dynamics: 
\[ \dot{e}_x = A_Le_x + BL\tilde{\theta}^T x_m \quad A_L = A + LC \]
\[ G(s) = S_1C(sI - A_L)^{-1}B \]

**Theorem** Given the above error dynamics, if \( G(s) \) is a strictly positive real transfer matrix then the proposed update law
\[ \dot{\theta} = -\Gamma x_m (S_1e_y)^\top \text{sign}(\Lambda) \]
will ensure that the adaptive system is globally stable and \( e_x(t) \to 0 \) as \( t \to \infty \).
$S_1, L$ and the KYP Lemma

**KYP Lemma** Given the transfer matrix $G(s)$ with stabilizable and detectable realization $(A, B, C)$, where $A$ is asymptotically stable, then $G(s)$ is SPR if and only if $\exists P > 0, Q > 0$ such that

1. $PB = C^\top$  
2. $A^\top P + PA = -Q$

**Theorem** Given $G(s) = S_1 C (sI - A_L)^{-1} B$, then $G(s)$ is SPR if $S_1$ and $L$ exist such that satisfy

- $PB = (S_1 C)^\top$
- $A_L^\top P + P A_L < 0$

where $A_L = A + L C$

Remaining problem: choose $S_1, L$ so that $G(s)$ is SPR

* Narendra, K.S., and Annaswamy, A.M. *Stable Adaptive Systems* 1989
Output Feedback Adaptive Control

Finding $S_1$ and $L$

Choice of $S_1$, $L$: Guarantees $G(s)$ is SPR

$$G(s) = S_1 C (sI - A_L)^{-1} B$$

To satisfy (1) choose

$$S_1 = ((CB)^T CB)^{-1} (CB)^T$$

All solutions $P$ to (2) then given by

$$P = \bar{C}^T (\bar{C}B)^{-T} \bar{C} + B_\perp X B_\perp^T$$

Choose $L$ so that

$$(A + LC)^T (\bar{C}^T (\bar{C}B)^{-T} \bar{C} + B_\perp X B_\perp^T) + (\bar{C}^T (\bar{C}B)^{-T} \bar{C} + B_\perp X B_\perp^T)(A + LC) < 0$$

- Leads to a Bilinear Matrix Inequality in $X$ and $L$
- Fix $X = \gamma I$. The BMI is reduced to a Linear Matrix Inequality
- Solve the resulting LMI for $L$
- Optimize $\gamma$ using the baseline controller
Output Feedback Adaptive Control
GHV Longitudinal Example: Tuning L

- $X = \gamma I$; vary $\gamma$ and compute corresponding $L$
- Compare baseline output feedback design and state feedback design
- Determine optimal $\gamma$

![Block diagram and plots showing control loop, magnitude, and sensitivity](image-url)
Output Feedback Adaptive Control
GHV Longitudinal Example: Tuning L

• The reference model is a component of the baseline controller

• It is critical that the baseline controller is designed with good frequency domain properties
  – Achieve desired crossover frequency
  – Roll off loop gain at high frequencies
  – Avoid large gains
  – Attain good gain and phase margins
GHV Longitudinal Example:
6 DOF Nonlinear Model With Adaptive Control

- State feedback control results in unstart due to sensor bias.
- Output feedback control avoids unstart due to sensor bias.
GHV Lateral-Direction Example: 6 DOF Nonlinear Model With Adaptive Control

- State feedback control results in **unstart due to sensor bias**.
- Output feedback control **avoids unstart** due to sensor bias.
Output Feedback Summary

• A novel adaptive controller has been developed that can **avoid unstart** in the presence of corrupted sensors

• **Ensures trajectories in the started region**; all others gravitate to unstart

• Main features
  – **Output feedback** design
  – Use of **closed-loop reference model**

• **My contributions**
  – Computationally simple **LMI based approach**
  – **Avoids bias-triggered unstart** for commands that do not trigger unstart
  – Simultaneously provide a **robust baseline** controller
Future Work

• Investigate conditions which guarantee feasibility of the LMI given the structure of X
• Determine analytical relationships for how the tuning of X impacts L and baseline closed-loop margins
• Examine how the selection of S1 impacts L and baseline closed-loop margins
Thank You

Questions?
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Recent contributions
SM contributions
### Aircraft Model

**Representation of Uncertainties: Loss of Control Effectiveness**

- Plant inputs are **aerodynamic control surfaces**
  - **Surfaces deflect** altering forces to create moments about CG

\[
\Delta C_L = \left( \frac{S_t - \Delta S_t}{S} \right) \eta \frac{dC_{Lt}}{d\delta_e} \delta_e
\]

- Control surfaces can become **damaged** during flight
- This modifies the underlying dynamics as

\[
\dot{x}_p = A_p x_p + B_p \Lambda u
\]

**Control effectiveness**
Aircraft Model

Representation of Uncertainties: Stability Derivatives

• **Stability derivative:** measure of how aerodynamic forces and moments change with state

Directional stability: $\Delta N_\beta$

Roll damping: $\Delta L_p$

\[
\begin{bmatrix}
\dot{\beta} \\
\dot{\phi}
\end{bmatrix}
= \begin{bmatrix}
Y_\beta & 0 & -1 & \frac{g}{U_{eq}} \\
L_\beta & \Delta L_p & L_r & 0 \\
\Delta N_\beta & N_p & N_r & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\beta \\
p \\
r \\
\phi
\end{bmatrix}
+ \begin{bmatrix}
0 & Y_{\delta_r} \\
L_{\delta_a} & L_{\delta_r} \\
N_{\delta_a} & N_{\delta_r} \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\delta_a \\
\delta_r
\end{bmatrix}
\]

• This linear representation of the aircraft dynamics with **uncertain stability derivatives** can be expressed

\[
\dot{x}_p = (A_p + B_p \Lambda W_p^\top)x_p + B_p u
\]

Stability derivative uncertainty
• Simplified longitudinal equations

\[ \dot{W} = QU + g_D + \frac{Z}{m} \]

\[ \dot{Q} = M_{yy} \]

• Moment about new CG location:

\[ M_\Delta (\alpha, q, \delta_e) = M_{\text{nom}} - L \Delta x \cos \alpha - D \Delta x \sin \alpha \]

\( \Delta x: \) Parametric uncertainty
Aircraft Model

Representation of Uncertainties: Longitudinal CG Shift

• Performing a Taylor Series expansion on Z and M

\[
M_{\Delta} = M_{\Delta eq} + \frac{\partial M_{\Delta}}{\partial \alpha} \alpha + \frac{\partial M_{\Delta}}{\partial q} q + \frac{\partial M_{\Delta}}{\partial \delta_e} \delta_e
\]

When CG shifts

\[
\dot{\alpha} = \frac{Z_{\alpha}}{V_{eq}} \alpha + \left(1 + \frac{Z_q}{V_{eq}}\right) q
\]

\[
\dot{q} = w_1 M_{\alpha} \alpha + w_2 M_q q + \lambda M_{\delta_e} \delta_e
\]

Unknown coefficients which scale the values of the aerodynamic and control coefficients

• This linear representation of the aircraft dynamics with uncertainty due to CG shift can be expressed as

\[
\dot{x}_p = (A_p + B_p \Lambda W_p^T) x_p + B_p \Lambda u
\]

• Changes control effectiveness and stability derivatives
Consider linear **nominal** plant

\[
\dot{x}_p = (A_p + B_p \Lambda W_p^\top)x_p + B_p \Lambda u
\]

**Step 1: Baseline control design**
- Decoupled longitudinal, lateral-directional, velocity dynamics: **design three control subsystems**
- Use **nominal** plant models

\[
\dot{x}_p = A_p x_p + B_p u
\]

- LQR + integral action: plant and control

\[
\dot{x} = Ax + Bu + B_{ref} \hat{z}_{cmd}
\]

\[
u_{bl} = K_{lqr}^\top x
\]
Control Design
Adaptive (Continued)

Adaptive control:
\[ u = (K_{lqr} + \theta(t))^\top x \]
\[ \theta(t) = f(e, x, \Gamma) \]

Error generation:
\[ \dot{x}_m = (A + BK_{lqr}^\top)x_m + B_{refz\text{cmd}} - L(x - x_m) \]
\[ e_x = x - x_m \]

Adjust \( \theta \) as:
\[ \dot{\theta} = -\Gamma x e_x^\top PB\text{sign}(\Lambda) \]
Adaptive Controller

Stability*

Plant:
\[ \dot{x} = (A + B\Lambda W^T)x + B\Lambda u + B_{ref}z_{cmd} \]

Closed-loop reference model:
\[ \dot{x}_m = (A + BK_{lqr}^T)x_m + B_{ref}z_{cmd} - L(x - x_m) \]

Errors:
\[ e_x = x - x_m \]
\[ \tilde{\theta} = \theta - \theta^* \]

Adaptive controller:
\[ u = (K_{lqr} + \theta(t))^T x \]
\[ \dot{\theta} = -\Gamma x e_x^T P B \text{sign}(\Lambda) \]

**Theorem:** the above system is globally stable, and \( e_x(t) \to 0 \) as \( t \to \infty \)

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Classical Adaptive Control
A Basis For Comparison (ORM-Method)

Plant: \[ \dot{x} = (A + B\Lambda W^\top)x + B\Lambda u + B_{\text{ref}}z_{\text{cmd}} \]

Closed-loop reference model: \[ \dot{x}_m = (A + BK_{\text{lqr}}^\top)x_m + B_{\text{ref}}z_{\text{cmd}} \]

Errors: \[ e_x = x - x_m \quad \tilde{\theta} = \theta - \theta^* \]

Adaptive controller: \[ u = (K_{\text{lqr}} + \theta(t))^\top x \]
\[ \dot{\theta} = -\Gamma x e_x^\top PB\text{sign}(\Lambda) \]

**Theorem:** the above system is globally stable, and \( e_x(t) \to 0 \) as \( t \to \infty \)
Validation of Adaptive Control

State Feedback Results: A Sample

- Angle of Attack
- Elevator Deflection Angle
- Normal Acceleration
- Roll Angle
- Aileron Deflection Angle
- Rudder Deflection Angle
State Feedback Summary*
Baseline + Adaptive Controller

• Successful tracking with adaptive controller
  – Task 1: 3 degree angle of attack doublet
  – Task 2: 80 degree roll step

• Tasks performed in the presence of following uncertainties
  a) Control ineffectiveness up to 50%
  b) CG shift up to -1.6 feet (11% of vehicle length)
  c) Stability derivative uncertainty up to 4x the nominal value
  d) Sensor bias in sideslip measurement 3.2 degrees
  e) Delay margin of 15-41 ms

Thank You

Questions?
Air Data Sensors
Measuring Incidence Angles: Traditional Aircraft

- Typical aircraft have “vane” or “probe” type sensor for measuring angle of attack and angle of sideslip.

- Such sensors protruding into a hypersonic flow would burn up.

Need to measure incidence angles using another method.
Air Data Sensors
Measuring Incidence Angles: Hypersonic Vehicles

• Can use a modification of the “probe” type sensor
  – Modified null-pressure sensor
  – Used on X-15
  – Works up to ~ Mach 5

• Alternatives
  – Optical sensors
Aircraft Model
Equations of Motion

- The equations of motion are given by the following

\[
\frac{d}{dt}(mV) = F \\
\frac{d}{dt}(J\omega) = M
\]

Contains forces and moments due to aerodynamics, thrust, and gravity

- The aerodynamic forces and moments are functionals that depend on aircraft state: orientation, translational and angular velocity, and control inputs

- These equations can be simplified and linearized

- The linearized aircraft dynamics form a decoupled set of equations
  - The velocity, longitudinal, and lateral-directional dynamics are independent from each other
  - Timescale separation used to further simplify equations
Aircraft Model
Obtaining the Linear Equations of Motion

1) Derive using Newton’s second law the translational and rotational acceleration of the aircraft ➔ **Nonlinear equations of motion**
   - **Simplify moment of inertia** using aircraft symmetry

2) **Linearize** the equations of motion about a steady flight condition
   - Simplify the **forces and moments** in the linearized equations of motion
     - Use known functional relationship for forces and moments ➔ Equations of motion will **decouple**
   - Use **timescale separation** to simplify further

**Result:** simplified, decoupled, linear equations of motion
Aircraft Model
Nonlinear Equations of Motion

\[ \dot{U} = RV - QW - g \sin(\Theta) + X/m \]
\[ \dot{V} = -RU + PW + g \sin(\Phi) \cos(\Theta) + Y/m \]
\[ \dot{W} = QU - PV + g \cos(\Phi) \cos(\Theta) + Z/m \]

\[ \dot{\Phi} = P + \tan(\Theta)[Q \sin(\Phi) + R \cos(\Phi)] \]
\[ \dot{\Theta} = Q \cos(\Phi) - R \sin(\Phi) \]
\[ \dot{\Psi} = [Q \sin(\Phi) + R \cos(\Phi)] / \cos(\Theta) \]

Component-wise equations derived assuming:

- Flat earth
- Aircraft is a rigid body
- Following products of inertia are zero:
  \[ J_{xy} = J_{yz} = 0 \]

Forces and moments contain aerodynamic and propulsive contributions.

\[ \dot{P} = \frac{J_{xz}(J_{xx} - J_{yy} + J_{zz})PQ + [J_{zz}(J_{yy} - J_{zz}) - J_{xz}^2]QR}{J_{xx}J_{zz} - J_{xz}^2} + \frac{J_{zz}L + J_{xz}N}{J_{xx}J_{zz} - J_{xz}^2} \]
\[ \dot{Q} = \frac{(J_{zz} - J_{xx})PR - J_{xz}(P^2 - R^2)}{J_{yy}} + \frac{M}{J_{yy}} \]
\[ \dot{R} = \frac{[(J_{xx} - J_{yy})J_{xx} + J_{xz}^2]PQ - J_{xz}[J_{xx} - J_{yy} + J_{zz}]QR}{J_{xz}^2 - J_{xx}J_{zz}} + \frac{J_{xz}L + J_{xx}N}{J_{xx}J_{zz} - J_{xz}^2} \]
Aircraft Model
Linearizing of Equations of Motion: Stability Derivatives

• Linearizing the equations of motion using the following perturbations about equilibrium

\[
\begin{align*}
U &= U_{eq} + u \\
V &= V_{eq} + v \\
W &= W_{eq} + w \\
P &= P_{eq} + p \\
Q &= Q_{eq} + q \\
R &= R_{eq} + R \\
\Phi &= \Phi_{eq} + \phi \\
\Theta &= \Theta_{eq} + \theta \\
\Psi &= \Psi_{eq} + \psi
\end{align*}
\]

• For longitudinal dynamics, assume functional dependency

\[M = M(\alpha, q, \delta_e)\]

• Taking partial derivatives yields the stability derivatives

\[
\begin{align*}
M_\alpha &= \frac{1}{J_{yy}} \frac{\partial M}{\partial \alpha} \\
M_q &= \frac{1}{J_{yy}} \frac{\partial M}{\partial q} \\
M_{\delta_e} &= \frac{1}{J_{yy}} \frac{\partial M}{\partial \delta_e}
\end{align*}
\]

• Repeating for y-body force Z gives

\[
\begin{bmatrix}
\dot{\alpha} \\
\dot{q}
\end{bmatrix} = \begin{bmatrix}
\frac{Z_\alpha}{V_0} & 1 + \frac{Z_q}{V_0} \\
M_\alpha & M_q
\end{bmatrix} \begin{bmatrix}
\alpha \\
q
\end{bmatrix} + \begin{bmatrix}
0 \\
M_{\delta_e}
\end{bmatrix} \delta_e
\]

Same process to linearize all equations
Aircraft Model

Linear Lateral-Directional Equations of Motion

- Consider the **lateral-directional** equations of motion

\[
\begin{bmatrix}
\dot{\beta} \\
\dot{p} \\
\dot{r} \\
\dot{\phi}
\end{bmatrix} =

\begin{bmatrix}
Y_\beta & 0 & -1 & \frac{g}{U_{eq}} \\
L_\beta & L_p & L_r & 0 \\
N_\beta & N_p & N_r & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\beta \\
p \\
r \\
\phi
\end{bmatrix} +

\begin{bmatrix}
0 & Y_{\delta_r} \\
L_{\delta_a} & L_{\delta_r} \\
N_{\delta_a} & N_{\delta_r} \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\delta_a \\
\delta_r
\end{bmatrix}
\]

- With **integral action** augmentation and **control effectiveness uncertainty**, these equations can be expressed as

\[
\dot{x} = Ax + B\Lambda u + B_{\text{ref}}z_{\text{cmd}}
\]

- Where the state vector contains
  - $\beta$ - Sideslip angle
  - $p$ - Roll rate
  - $q$ - Pitch rate
  - $\phi$ - Roll angle
Aircraft Model

Representation of Uncertainties: Longitudinal CG Shift

• Linearize the pitch rate equation

\[ \dot{Q} = \frac{M}{J_{yy}} \]

\[ \dot{q} = \frac{1}{J_{yy}} \frac{\partial M}{\partial \alpha} \alpha + \frac{1}{J_{yy}} \frac{\partial M}{\partial q} q + \frac{1}{J_{yy}} \frac{\partial M}{\partial \delta_e} \delta_e \]

\[ \dot{q} = M_\alpha \alpha + M_q q + M_{\delta_e} \delta_e \]

• When the CG has been shifted, we have shown that M becomes M_Delta

• Taking the partial derivatives of M_Delta we can write

\[ \frac{\partial M_\Delta}{\partial \alpha} = w_1 \frac{\partial M}{\partial \alpha} \quad \frac{\partial M_\Delta}{\partial q} = w_2 \frac{\partial M}{\partial q} \quad \frac{\partial M_\Delta}{\partial \delta_e} = \lambda \frac{\partial M}{\partial \delta_e} \]

• Giving

\[ \dot{q} = w_1 M_\alpha \alpha + w_2 M_q q + \lambda M_{\delta_e} \delta_e \]
The sensitivity analysis seeks to determine whether a change in the initial condition of a single state will more strongly influence some modes than others. The modal matrix $M$

\[ M = [v_1 \ v_2 \ \ldots \ \ v_n] \]

Denote the rows of $M$ as $r_1, \ldots, r_n$ and the columns as $c_1, \ldots, c_n$. Each column vector $c_i$ has components

\[ c_i = \begin{bmatrix} c_{1i} \\ \vdots \\ c_{ni} \end{bmatrix} \quad C_i = \begin{bmatrix} c_{1i} & 0 & 0 & \ldots & 0 \\ 0 & c_{2i} & 0 & \ldots & 0 \\ \vdots & & & \ddots & 0 \\ 0 & 0 & 0 & \ldots & c_{ni} \end{bmatrix} \]

Calculate $S$ as

\[ S = \begin{bmatrix} r_1 C_1 \\ r_2 C_2 \\ \vdots \\ r_n C_n \end{bmatrix} \]

where each row of $S$ corresponds to a state, and each column corresponds to a mode.
# Decoupled Dynamics

## Sensitivity Matrix

<table>
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<tr>
<th>VT</th>
<th>λ₁</th>
<th>λ₂</th>
<th>λ₃</th>
<th>λ₄</th>
<th>λ₅</th>
<th>λ₆</th>
<th>λ₇</th>
<th>λ₈</th>
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<td>6.57E-05</td>
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<td>2.46E-09</td>
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<td>0.0044</td>
<td>8.55E-10</td>
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<td>β</td>
<td>1.79E-10</td>
<td>0.2311</td>
<td>1.16E-07</td>
<td>0.3844</td>
<td>0.3844</td>
<td>3.17E-11</td>
<td>3.17E-11</td>
<td>3.30E-15</td>
<td>7.59E-05</td>
</tr>
<tr>
<td>p</td>
<td>2.81E-09</td>
<td>0.4259</td>
<td>5.66E-08</td>
<td>0.2855</td>
<td>0.2855</td>
<td>7.34E-10</td>
<td>7.34E-10</td>
<td>4.73E-12</td>
<td>0.0031</td>
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<tr>
<td>r</td>
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<td>0.0119</td>
<td>9.56E-09</td>
<td>0.3412</td>
<td>0.3412</td>
<td>7.91E-09</td>
<td>7.91E-09</td>
<td>1.73E-10</td>
<td>0.3058</td>
</tr>
<tr>
<td>φ</td>
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<td>0.0237</td>
<td>4.84E-08</td>
<td>0.3096</td>
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<td>1.08E-08</td>
<td>1.08E-08</td>
<td>2.88E-09</td>
<td>0.3570</td>
</tr>
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</table>
Open Loop Poles

![Graph showing poles on a complex plane with labels for Dutch Roll, Phugoid, Spiral, and Short Period.](image-url)
Control Design
Baseline: Integral Error Augmented State Space Representation

Consider linear **nominal** plant with regulated output $z$

$$\dot{x}_p = A_p x_p + B_p u$$
$$z = C_{zp} x_p$$

Introduce **integral error state**

$$\dot{x}_e = z_{cmd} - z$$

Augmenting the state space description

$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_e \end{bmatrix} = \begin{bmatrix} A_p & 0 \\ -C_{pz} & 0 \end{bmatrix} \begin{bmatrix} x_p \\ x_e \end{bmatrix} + \begin{bmatrix} B_p \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ I \end{bmatrix} z_{cmd}$$

Using $x = [ x_p^T \ x_e^T ]^T$ the state space description can be written

$$\dot{x} = Ax + Bu + B_{ref} z_{cmd}$$

where

$$\begin{bmatrix} A_p & 0 \\ -C_{pz} & 0 \end{bmatrix} \quad B = \begin{bmatrix} B_p \\ 0 \end{bmatrix} \quad B_{ref} = \begin{bmatrix} 0 \\ I \end{bmatrix}$$
Control Design
Uncertain Integral Augmented State Space Description

Consider linear **uncertain** plant

\[
\dot{x}_p = (A_p + B_p \Lambda W_p^\top)x_p + B_p \Lambda u
\]

Augmenting with **integral error state**

\[
\begin{bmatrix}
\dot{x}_p \\
\dot{x}_e
\end{bmatrix} = \begin{bmatrix}
A_p & 0 \\
-C_{pz} & 0
\end{bmatrix}
\begin{bmatrix}
x_p \\
x_e
\end{bmatrix} + \begin{bmatrix}
B_p \Lambda W_p^\top & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x_p \\
x_e
\end{bmatrix} + \begin{bmatrix}
B_p \\
0
\end{bmatrix} \Lambda u + \begin{bmatrix}
0 \\
I
\end{bmatrix} z_{cmd}
\]

Using \( W^\top = \begin{bmatrix} W_p^\top & 0_{m \times n_e} \end{bmatrix} \) this can be written

\[
\begin{bmatrix}
\dot{x}_p \\
\dot{x}_e
\end{bmatrix} = \begin{bmatrix}
A_p & 0 \\
-C_{pz} & 0
\end{bmatrix}
\begin{bmatrix}
x_p \\
x_e
\end{bmatrix} + \begin{bmatrix}
B_p \\
0
\end{bmatrix} \Lambda W^\top
\begin{bmatrix}
x_p \\
x_e
\end{bmatrix} + \begin{bmatrix}
B_p \\
0
\end{bmatrix} \Lambda u + \begin{bmatrix}
0 \\
I
\end{bmatrix} z_{cmd}
\]

More concisely

\[
\dot{x} = (A + B \Lambda W^\top)x + B \Lambda u + B_{ref} z_{cmd}
\]
Stability

Given \( \{A_L, B, S_1C\} \) is SPR, we can show stability by proposing the following candidate Lyapunov function

\[
V(e_x, \tilde{\theta}) = e_x^\top P e_x + \text{tr}(|\Lambda|\tilde{\theta}^\top \Gamma^{-1} \tilde{\theta})
\]

Using the following update law

\[
\dot{\tilde{\theta}} = -\Gamma x_m (S_1 e_y)^\top \text{sgn}(\Lambda)
\]

the time derivative of \( V \) along system trajectories can be evaluated as

\[
\dot{V} = e_x^\top P e_x + e_x^\top P \dot{e}_x + \text{tr}(|\Lambda|\tilde{\theta}^\top \Gamma^{-1} \tilde{\theta}) + \text{tr}(|\Lambda|\tilde{\theta}^\top \Gamma^{-1} \dot{\tilde{\theta}}) = -e_x^\top Q e_x
\]

Thus, \( \dot{V}(e_x, \tilde{\theta}) \leq 0 \). Given that \( V > 0 \) and \( \dot{V} \leq 0 \) we have \( V(e_x(t), \tilde{\theta}(t)) \leq V(e_x(0), \theta(0)) < \infty \). Thus \( V \) is bounded, and so \( e \) and \( \tilde{\theta} \) are bounded. Since \( z_{\text{cmd}} \) is bounded and the reference model is stable, \( x_m \) is bounded, giving \( x \) is bounded. This can be compactly stated as \( e, x, \tilde{\theta} \in \mathcal{L}_\infty \).
Next, note that

$$\int_0^t \dot{V}(e_x(t), \tilde{\theta}(t)) dt = V(e(t), \tilde{\theta}(t)) - V(e_x(0), \tilde{\theta}(0))$$

From this we have

$$\int_0^t e_x(t)^T Q e_x(t) dt \leq V(e_x(0), \tilde{\theta}(0))$$

And with $e_x^T \lambda_{\min}(Q)e_x \leq e_x^T Q e_x$ we have $\int_0^t \|e_x(t)\|_2^2 dt \leq \frac{V(0)}{\lambda_{\min}(Q)}$ giving $e_x \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. Furthermore, $\dot{e}_x \in \mathcal{L}_\infty$ and so Barbalat’s lemma is satisfied and therefore $\lim_{t \to \infty} e_x(t) = 0$.

Lastly, since $e_x \in \mathcal{L}_2$, $x_m(t) \to x^o_m(t)$ as $t \to \infty$, where $x^o_m$ is the state of the “fixed target”, or open-loop reference model when $L = 0$. 
Barbalat’s Lemma: If the differentiable function $f(t)$ has a finite limit as $t \to \infty$, and if $\dot{f}$ is uniformly continuous, then $\dot{f}(t) \to 0$ as $t \to \infty$.

A sufficient condition for a differentiable function to be uniformly continuous is that its derivative is bounded.
Existence of $S_1$ and $L$

The following must hold

1. $(A,B)$ is controllable and $(A,C)$ is observable
2. $\text{Rank}(B)=m$ (full rank)
3. $\text{Rank}(CB)=p$
4. $(A,B,C)$ has only minimum phase transmission zeros
Definitions

Definition **Asymptotically Stable Matrix**: Matrix with all eigenvalues strictly in left half plane.

Definition **Strictly Positive Real Transfer Matrix**: A square transfer matrix \( G(s) \) is PR if

1. Each element of \( G(s) \) has poles not in RHP
2. \( G^*(s) = G(s^*) \) for strict RHP poles \( \text{Re}[s] > 0 \)
3. \( G^\top(s^*) + G(s) \geq 0 \) for strict RHP poles \( \text{Re}[s] > 0 \)

\( G(s) \) is SPR if it is PR for \( G(s - \epsilon) \) for some \( \epsilon > 0. \)

Definition **Stabilizable**: System \((A, B)\) is stabilizable if all of the unstable modes are controllable

Definition **Detectable**: System \((A, C)\) is detectable if all of the unstable modes are observable